A class of abstract delay differential equations
in the light of suns and stars

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Outline

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Abstract delay differential equations

We are interested in the initial value problem

\[ \dot{x}(t) = Bx(t) + F(x_t), \quad t \geq 0, \]  
\[ x_0 = \varphi \in X, \]  

where

- \( Y \) is an arbitrary real or complex Banach space,
- \( B \) generates a \( C_0 \)-semigroup of linear operators on \( Y \),
- \( X := C([-h, 0], Y) \) is the state space,
- \( x_t \in X \) is the history at time \( t \geq 0 \),

\[ x_t(\theta) := x(t + \theta), \quad \forall \theta \in [-h, 0], \]

- \( F : X \to Y \) is continuous and may be nonlinear.

We call (DDE) an abstract DDE with initial condition (IC).
Initial condition $\varphi$

$x(t) \in Y$

History $x_t$ at time $t$
Examples of existing work

There is a substantial literature on the semilinear problem:

Using formal dualities [Hale, Verduyn Lunel]:

- [Travis and Webb, 1974],
- [Wu, 1996],
- [Faria, Huang and Wu, 2002],
- [Faria, 2006].

These works assume that $S$ is compact.
If $\dim Y = \infty$ then compactness of $S$ excludes $B = 0$.

Purely linear theory in spaces of continuous or integrable functions:

- [Engel and Nagel, 2000],
- [Bátkai and Piazzera, 2005].
Aim of the present work

1. Establish one-to-one correspondence between (DDE, IC) and

   \[ u(t) = T_0(t)\varphi + j^{-1} \int_0^T T_0^{\circ,*}(t - \tau)G(u(\tau)) \, d\tau \]  

   (AIE)

   for an appropriate $C_0$-semigroup $T_0$ on $X$ and a continuous nonlinear perturbation $G : X \to X^{\circ,*}$.

2. Study bounded linear perturbations of an arbitrary $C_0$-semigroup on an arbitrary non-sun-reflexive Banach space.

3. Study nonlinear Lipschitz continuous perturbations and verify existence and properties of local invariant manifolds.

4. Apply general results to the particular class of abstract DDEs.
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The sun-star skeleton

\[ C([-h, 0], Y) \quad \xrightarrow{X} \quad \text{NBV}([0, h], Y^*) \]

\[ X \quad \xrightarrow{j} \quad X^{\circ\circ} \]

\[ X^{\circ\circ} \quad \xleftarrow{X^{\circ\circ\circ}} \quad X^{\circ\ast} \]

\[ X^{\circ\ast} \quad \xleftarrow{j} \quad X^{\circ} \]

Now specify \( T_0 \) on \( X \) and find a representation of the sun dual \( X^{\circ} \).
Specification of $T_0$ on $X$

Inspired by classical DDEs, consider the ‘trivial’ initial value problem

\[
\begin{aligned}
\dot{x}(t) &= Bx(t), \quad t \geq 0, \\
x_0 &= \varphi \in X,
\end{aligned}
\]

for (DDE) with $F = 0$.

It has the explicit mild solution $x^{\varphi} : [-h, \infty) \to Y$ given by

\[
x^{\varphi}_0 = \varphi, \quad x^{\varphi}(t) = S(t)\varphi(0), \quad t \geq 0.
\]

Define the strongly continuous shift semigroup $T_0$ on $X$ by

\[
T_0(t)\varphi := x^{\varphi}_t,
\]

for all $\varphi \in X$ and $t \geq 0$. 
A representation theorem for $X^\circ$

Define $\chi_0 : [0, h] \to \{0, 1\}$ by $\chi_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$

Theorem

The maximal subspace of strong continuity of $T_0^*$ is

$$X^\circ = \left\{ f : [0, h] \to Y^* : \text{there exist } y^\circ \in Y^\circ \text{ and } g \in L^1([0, h], Y^*) \right\}$$

such that $f(t) = \chi_0(t)y^\circ + \int_0^t g(s) \, ds$,

and $\iota : Y^\circ \times L^1([0, h], Y^*) \to X^\circ$ defined by

$$\iota(y^\circ, g)(t) := \chi_0(t)y^\circ + \int_0^t g(s) \, ds, \quad \forall t \in [0, h],$$

is an isometric isomorphism.
The proof uses properties of the \textit{bilinear Riemann-Stieltjes integral}.

\textbf{Proof.}

1. Denote by $E \subseteq X^*$ the candidate subspace for $X^\ominus$.
2. Find a representation for $T_0^*$ on $E$.
3. For arbitrary $f \in E$ show that

$$|\langle \varphi, T_0^*(t)f \rangle - \langle \varphi, f \rangle| \to 0, \quad t \downarrow 0,$$

uniformly for $\|\varphi\| = 1$. This proves $E \subseteq X^\ominus$.
4. $E$ is closed because $\iota$ is a linear isometry into $X^*$.
5. Show that $R(\lambda, A_0^*)$ maps $X^*$ into $E$.
6. Take closures on both sides of $\mathcal{D}(A_0^*) \subseteq E$. \hfill \Box

The resolvent trick was inspired by [Greiner and Van Neerven, 1992].
From now on, we identify \( X^\circ \simeq Y^\circ \times L^1([0, h], Y^*) \).

**Theorem**

The duality pairing between \( \varphi \in X \) and \( \varphi^\circ = (y^\circ, g) \in X^\circ \) is

\[
\langle \varphi, \varphi^\circ \rangle = \langle \varphi(0), y^\circ \rangle + \int_0^h \langle \varphi(-\theta), g(\theta) \rangle \, d\theta.
\]

For the action of \( T_0^\circ \) on \( \varphi^\circ = (y^\circ, g) \in X^\circ \) we have

\[
T_0^\circ(t) \varphi^\circ = (S^\circ(t)y^\circ + \int_0^{t \wedge h} S^*(t - \theta)g(\theta) \, d\theta, T_1(t)g),
\]

where \( T_1 \) is translation on \( L^1([0, h], Y^*) \) and the integral is a weak* Lebesgue integral with values in \( Y^\circ \).
Dressing the sun-star skeleton

\[ T_0 \text{ on } C([-h, 0], Y) \quad \text{to} \quad \text{NBV}([0, h], Y^*) \]

\[ \text{Diagram:} \]

\[ X \quad \xrightarrow{\text{j}} \quad X^* \]

\[ Y^{\otimes*} \times [L^1([0, h], Y^*)]^* \quad \xrightarrow{\text{to}} \quad Y^\otimes \times L^1([0, h], Y^*) \]
We have the identification

\[
[L^1([0, h], Y^*)]^* \simeq L^\infty([-h, 0], Y^{**})
\]

if and only if \( Y^{**} \) has the Radon-Nikodým property.

So, in general we can only write

\[
X^{\odot*} \simeq Y^{\odot*} \times [L^1([0, h], Y^*)]^*.
\]

Still, \( Y^{\odot*} \times L^\infty([-h, 0], Y^{**}) \) is isometrically embedded in \( X^{\odot*} \). Relevant for computations, for example in bifurcation theory.
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The lack of sun-reflexivity

If $Y$ is not sun-reflexive for $S$ then $X$ is not sun-reflexive for $T_0$.

For $u : \mathbb{R}_+ \to X$ continuous, the weak* integral in (AIE)

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0^\circ\varphi(t - \tau)G(u(\tau)) \, d\tau$$

takes values in $X^{\circ\circ}$. If $X$ is not sun-reflexive for $T_0$ then

$$j(X) \subset X^{\circ\circ}$$

with strict inclusion, so it is not clear if $j^{-1}$ can be applied.
The range of the weak* integral

Recall that we have proven that $X^{\odot*} \simeq Y^{\odot*} \times [L^1([0, h], Y^*)]^*$. Inspired by classical DDEs, define

$$\ell : Y \rightarrow X^{\odot*}, \quad \ell y := (j_Y y, 0).$$

Proposition

Let $f : \mathbb{R}_+ \rightarrow Y$ be continuous and let $t \geq 0$. Then

$$\int_0^t T_0^{\odot*}(t - \tau) \ell f(\tau) d\tau = j\psi,$$

where $\psi \in X$ is given by

$$\psi(\theta) := \int_0^{(t+\theta)^+} S(t - \tau + \theta)f(\tau) d\tau, \quad \forall \theta \in [-h, 0].$$

So, the weak* integral takes values in the range of $j$. 
Proof.

1. Observe that \( \ell \) has a ‘pre-adjoint’. If we define

\[
\delta : X^\circ \rightarrow Y^\circ, \quad \delta(y^\circ, g) := y^\circ,
\]

then \( \delta^* : Y^{\circ*} \rightarrow X^{\circ*} \) and \( \ell = \delta^* j_Y \).

2. Recall that we found a representation for \( T_0^\circ \).

3. Use it to represent \( T_0^{\circ*} \) on the range of \( \ell \).

4. Check that

\[
\langle \varphi^\circ, \int_0^t T_0^{\circ*}(t - \tau)\ell f(\tau) \, d\tau \rangle = \langle \psi, \varphi^\circ \rangle, \quad \forall \varphi^\circ \in X^\circ
\]

also using the duality pairing between \( X \) and \( X^\circ \). \( \square \)
Corollary

Let $T_0$ be the shift semigroup, $F : X \to Y$ Lipschitz continuous and $G := \ell \circ F$. For every initial condition $\varphi \in X$ there exists a unique continuous function $u : \mathbb{R}_+ \to X$ that satisfies

$$u(t) = T_0(t) \varphi + j^{-1} \int_0^t T_0^{\circ*}(t - \tau) G(u(\tau)) \, d\tau,$$

i.e. $u$ is the global solution of $(\text{AIE})$.

As usual, a local Lipschitz condition gives local solutions that are unique in the maximal sense.
Correspondence between (DDE, IC) and (AIE)

We return to (DDE, IC),
\[
\dot{x}(t) = Bx(t) + F(x_t), \quad t \geq 0, \quad \text{(DDE)}
\]
\[
x_0 = \varphi \in X, \quad \text{(IC)}
\]
to make the connection with (AIE).

Definition
A continuous function \( x : [-h, \infty) \to Y \) that satisfies \( x_0 = \varphi \) and
\[
x(t) = S(t)\varphi(0) + \int_{0}^{t} S(t - \tau)F(x_{\tau}) \, d\tau, \quad \forall \, t \geq 0,
\]
is called a mild solution of (DDE, IC).

Any classical solution is a mild solution.
If \( B = 0 \) then any mild solution is a classical solution.
As a consequence of the earlier proposition on the range of the weak* integral, we arrive at:

**Theorem**

Let $T_0$ be the shift semigroup, $F : X \to Y$ continuous and $G := \ell \circ F$. Let $\varphi \in X$ be an initial condition.

1. Suppose $x$ is a mild solution of $(\text{DDE, IC})$. Define $u : \mathbb{R}_+ \to X$ by
   \[
   u(t) := x_t, \quad \forall t \geq 0.
   \]
   Then $u$ is a solution of $(\text{AIE})$.

2. Suppose $u$ is a solution of $(\text{AIE})$. Define $x : [-h, \infty) \to Y$ by
   \[
   x_0 := \varphi, \quad x(t) := u(t)(0), \quad t \geq 0.
   \]
   Then $x$ is a mild solution of $(\text{DDE, IC})$. 
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Given this correspondence, standard results from ‘the book’, such as
• the principle of linearized stability,
• existence and smoothness of local invariant manifolds,
• local bifurcation theorems,
are expected to remain valid for abstract DDEs.

To make this rigorous, we need to check in detail
• what happens without sun-reflexivity,
• if weaker conditions are sufficient. (Yes, so far.)

Maybe this could also be relevant for twin semigroups?
[Diekmann and Verduyn Lunel, forthcoming, 2019]
Aim of the forthcoming work

1. Establish one-to-one correspondence between (DDE, IC) and

\[ u(t) = T_0(t) \varphi + j^{-1} \int_0^t T_0^{\circ*}(t - \tau) G(u(\tau)) \, d\tau \quad \text{(AIE)} \]

for an appropriate \( C_0 \)-semigroup \( T_0 \) on \( X \) and a continuous nonlinear perturbation \( G : X \to X^{\circ*} \).

2. Study bounded linear perturbations of an arbitrary \( C_0 \)-semigroup on an arbitrary non-sun-reflexive Banach space.

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4. Apply general results to the particular class of abstract DDEs.
Motivating example
In [Van Gils, Janssens, Kuznetsov and Visser, 2013] an abstract DDE with $B = 0$ and non-reflexive $Y$ was considered. There,

- existence of a smooth local center manifold,
- differentiability of solutions of (AIE) that lie on it,

were already assumed to hold in the non-sun-reflexive case.

Regularity of $S$
We will require more than just strong continuity from $S$. If $S$ is immediately norm continuous, then $T$ is eventually norm continuous and

$$s(A) = \omega_0(T).$$
For further reading


A. Bátkai and S. Piazzera.  
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