Abstract delay equations

in the light of suns and stars

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Outline

Introduction and motivation

Admissibility

Maximality, robustness, and splitting

Consequences and conclusions

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Two classes of abstract delay equations

Given a Banach space Y and an initial history $\varphi : [-h, 0] \to Y$,

extend φ to $x : [-h, t_e) \to Y$ for some $0 < t_e \leq \infty$.

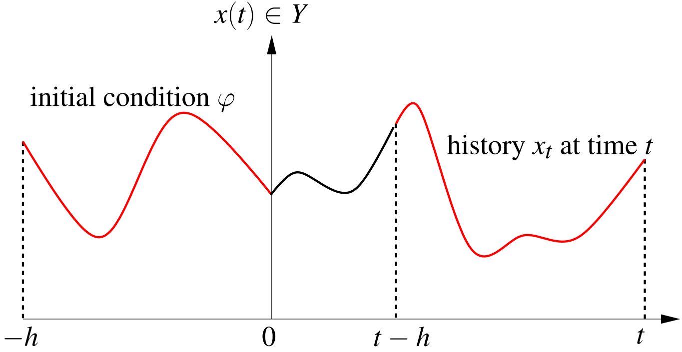
Two of many ways:

RE Prescribe $x(t)$ as function of x_t for $t > 0$.

DDE Prescribe derivative $\dot{x}(t)$ as function of x_t for $t \geq 0$.

Shift the extension on $[t - h, t]$ back to $[-h, 0]$.

Obtain a dynamical system on a state space X of histories.


$x(t) \in Y$
initial condition φ
history x_t at time t
$-h$
0
$t-h$
t

Adjoint semigroups for delay equations

TYPE	STATE SPACE X	DE	DE_0
RE	$L^p([-h,0],Y)$	$x(t) = F(x_t)$	$x(t) = 0$
DDE	$C([-h,0],Y)$	$\dot{x}(t) = Bx(t) + F(x_t)$	$\dot{x}(t) = Bx(t)$

$F : X \to Y$ is a continuous operator,

B generates a $\mathcal{C}_0$-semigroup S on Y.

Solutions of (DE_0 + IC) define a $\mathcal{C}_0$ shift semigroup T_0 on X.

Adjoint semigroup theory for T_0 on X gives a canonical embedding

$$j : X \to X^{\odot\star}, \qquad \langle x^{\odot}, jx \rangle := \langle x, x^{\odot} \rangle .$$

Perturbation of the w*-generator $A_0^{\odot\star}$ of $T_0^{\odot\star}$ with an operator $G : X \to X^{\odot\star}$ gives a semilinear differential equation in $X^{\odot\star}$,

$$d^\star (j \circ u)(t) = A_0^{\odot\star} ju(t) + G(u(t)),$$

suggesting an abstract integral equation in X,

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0^{\odot\star}(t-\tau) G(u(\tau))\, d\tau. \tag{IE}$$

For RE and DDE, in general T_0 is not sun-reflexive.

Still, there is a bounded embedding $\ell : Y \to X^{\odot\star}$ such that[1]

1. for all continuous $w : \mathbb{R}_+ \to Y$ and all $t \geq 0$,

$$\int_0^t T_0^{\odot\star}(t-\tau)\ell w(\tau)\, d\tau \in jX,$$

   and

2. for a given initial history $\varphi \in X$, solutions of (IE) with perturbation

$$G = \ell \circ F : X \to Y \to X^{\odot\star}$$

   are in bijection with solutions of (DE + IC).

[1] [Diekmann and Gyllenberg, 2008] for RE and [Janssens, 2019] for DDE.

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Definitions

Let T_0 be a $\mathcal{C}_0$-semigroup on a Banach space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Do not assume that T_0 is sun-reflexive.

Let J be a non-degenerate interval, and

$$\Omega_J := \{(t,s) \in J \times J \,:\, t \geq s\}.$$

Given a continuous function $f : J \to X^{\odot\star}$, study the range in $X^{\odot\star}$ of the convolution map

$$\Omega_J \ni (t,s) \mapsto \int_s^t T_0^{\odot\star}(t-\tau) f(\tau)\, d\tau \in X^{\odot\star}.$$

1. A continuous function $f : J \to X^{\odot\star}$ is admissible for T_0 if

$$\int_s^t T_0^{\odot\star}(t-\tau)f(\tau)\,d\tau \in jX \qquad \text{for all } (t,s) \in \Omega_J.$$

2. A closed subspace $\mathcal{X}_0$ of $X^{\odot\star}$ is an admissible range for T_0 if *every* continuous function $f : J \to \mathcal{X}_0$ is admissible for T_0. This is *independent* of the interval J.

3. A continuous $G : X \to X^{\odot\star}$ is an admissible perturbation for T_0 if G takes its values in some T_0-admissible range.

An admissibility test

Lemma

Let $\mathcal{X}_0$ be a closed subspace of $X^{\odot\star}$. If there exists an interval J such that every constant function on J into $\mathcal{X}_0$ is T_0-admissible, then $\mathcal{X}_0$ is a T_0-admissible range.

Proof.

0. We can assume that J is compact.
1. Linear functions on J into $\mathcal{X}_0$ are T_0-admissible.
2. The same is true for affine functions,
   and for continuous piecewise affine functions.
3. The latter function class is dense in $C(J, \mathcal{X}_0)$.
4. Uniform convergence preserves admissibility. □

Three questions about admissibility

maximality

Does there exist a maximal admissible range for T_0?

robustness

Let $\mathcal{X}_0$ be an admissible range for T_0.

Let T be obtained by perturbing T_0 with $L \in \mathcal{L}(X, \mathcal{X}_0)$.

Is $\mathcal{X}_0$ an admissible range for T as well?

splitting

Let $f : [0, t_e) \to \mathcal{X}_0$ be continuous, for some $0 < t_e \leq \infty$.

Is perturbing T by f equivalent to perturbing T_0 by $L + f$.

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Maximality. The subspace $X_0^{\odot\times}$

For $\lambda \in \mathbb{K}$ with $\operatorname{Re}\lambda$ sufficiently large, the resolvent

$$R(\lambda, A_0^{\odot\star}) := (\lambda I - A_0^{\odot\star})^{-1}$$

exists in $\mathcal{L}(X^{\odot\star})$. Define[2]

$$X_0^{\odot\times} := \{x^{\odot\star} \in X^{\odot\star} \,:\, R(\lambda, A_0^{\odot\star})x^{\odot\star} \in jX\}.$$

This does not depend on λ.
$X_0^{\odot\times}$ is closed and $T_0^{\odot\star}$-invariant, and
coincides with $X^{\odot\star}$ if and only if T_0 is sun-reflexive.

$X_0^{\odot\times}$ is instrumental in the next two theorems about admissibility[3].

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[2] [Van Neerven, 1992]
[3] [Janssens, 2020]

Theorem

$X_0^{\odot\times}$ *is an admissible range for* T_0 *that is maximal for inclusion*

Proof of admissibility.

0. Show that constant functions into $X_0^{\odot\times}$ are admissible for T_0 and apply testing lemma. For arbitrary $x^{\odot\times} \in X_0^{\odot\times}$,
1. Observe that $y_\lambda^{\odot\star} := R(\lambda, A_0^{\odot\star})x^{\odot\times}$ is in $\mathcal{D}(A_0^{\odot\star})$ and in jX.
2. Evaluate, for any $s \leq t$,

$$\int_s^t T_0^{\odot\star}(t-\tau)x^{\odot\times}\,d\tau = \int_s^t T_0^{\odot\star}(t-\tau)(\lambda I - A_0^{\odot\star})y_\lambda^{\odot\star}\,d\tau$$
$$= \lambda \int_s^t T_0^{\odot\star}(t-\tau)y_\lambda^{\odot\star}\,d\tau - (T_0^{\odot\star}(t-s) - I)y_\lambda^{\odot\star},$$

and note that the RHS sits in jX. □

Proof of maximality.

0. Let $\mathcal{X}_0$ be an admissible range for T_0. For arbitrary $x^{\odot\star} \in \mathcal{X}_0$,
1. Verify the adjoint Laplace transform representation

$$R(\lambda, A_0^{\odot\star})x^{\odot\star} = \lim_{t\to\infty} \int_0^t T_0^{\odot\star}(\tau)e^{-\lambda\tau}x^{\odot\star}\,d\tau, \qquad (\#)$$

with convergence in the norm of $X^{\odot\star}$.

2. Evaluate, for any $t \geq 0$,

$$\int_0^t T_0^{\odot\star}(\tau)e^{-\lambda\tau}x^{\odot\star}\,d\tau = e^{-\lambda t}\int_0^t T_0^{\odot\star}(t-\tau)e^{\lambda\tau}x^{\odot\star}\,d\tau,$$

and note that the RHS sits in jX.

3. Apply (#) and use norm-closedness of jX. □

Corollary

A continuous perturbation $G : X \to X^{\odot\star}$ *is admissible for* T_0 *if and only if* G *takes its values in* $X_0^{\odot\times}$.

Robustness and splitting

Theorem

Robustness

Let T be obtained by perturbing T_0 with $L \in \mathcal{L}(X, X_0^{\odot\times})$.

Then $X_0^{\odot\times}$ is an admissible range for T as well.

Splitting

Let $f : J \to X_0^{\odot\times}$ be continuous on a compact time interval J.

The unique solution $u : J \to X$ of

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0^{\odot\star}(t-\tau)[Lu(\tau) + f(\tau)]\, d\tau \qquad (\dagger)$$

is given by

$$u(t) = T(t)\varphi + j^{-1} \int_0^t T^{\odot\star}(t-\tau) f(\tau)\, d\tau.$$

Proof.

0. Suppose that $\varphi \in j^{-1}\mathcal{D}(A_0^{\odot\star})$ and $f : J \to X_0^{\odot\times}$ Lipschitz.
1. There exist Lipschitz $u_m : J \to X$ and $f_m : J \to X_0^{\odot\times}$ such that

$$u_m(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0^{\odot\star}(t-\tau)[Lu_m(\tau) + f_m(\tau)]\,d\tau,$$

   for all $t \in J$, and $f_m \to f$ and $u_m \to u_{\varphi,f}$ uniformly on J, where $u_{\varphi,f} : J \to X$ is the unique solution of (†).
2. Use the regularity of φ, u_m, and f_m to w*-differentiate and split,

$$\begin{aligned} d^\star(j \circ u_m)(t) &= A_0^{\odot\star} j u_m(t) + Lu_m(t) + f_m(t) \\ &= A^{\odot\star} j u_m(t) + f_m(t) \end{aligned}$$

   for $t \in J$, with $u_m(0) = \varphi$.

3. w*-integrate from arbitrary s to t in J,

$$ju_m(t) - jT(t-s)u_m(s) = \int_s^t T^{\odot\star}(t-\tau)f_m(\tau)\,d\tau,$$

hence f_m is admissible for T.

4. Let $m \to \infty$ uniformly on J to conclude f is T-admissible, and

$$u_{\varphi,f}(t) = T(t)\varphi + j^{-1}\int_0^t T^{\odot\star}(t-\tau)f(\tau)\,d\tau.$$

5. The general case for $\varphi \in X$ and $f \in C(J, X_0^{\odot\times})$ follows from the continuity of

$$X \times C(J, X_0^{\odot\times}) \ni (\varphi, f) \mapsto u_{\varphi,f} \in C(J, X)$$

and density of $j^{-1}\mathcal{D}(A_0^{\odot\star}) \times \mathrm{Lip}(J, X_0^{\odot\times})$. □

Corollary (of maximality and robustness)

The maximal admissible ranges for T and T_0 coincide: $X^{\odot\times} = X_0^{\odot\times}$.

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Implications for nonlinear local analysis

Do not insist on sun-reflexivity of X for T_0. Instead, systematically require all perturbations to take values in $X^{\odot\times}$.

Let $G : X \to X^{\odot\times}$ be C^k for $k \geq 1$ and $G(0) = 0$.

Variation-of-constants

$$u(t) = T(t-s)u(s) + j^{-1} \int_s^t T^{\odot\star}(t-\tau)R(u(\tau))\,d\tau, \qquad s \leq t,$$

is well-defined, with $L := DG(0)$ and $R := G - L$ into $X^{\odot\times}$.

This has led to relatively easy generalizations of sun-reflexive results, such as local center manifold theorems[4].

[4] Compare [Diekmann et. al., 1995] with [Janssens, 2020, Theorems 39 and 41]

Existing and new motivation

These theorems underlie bifurcation analysis in abstract DDE models[5],

$$\dot{x}(t) = Bx(t) + F(x_t), \qquad t \geq 0.$$

with S generated by B immediately norm-continuous on Y.

So the cases $B = 0$ and $B \neq 0$ are treated on an equal footing.

Recent motivation comes from

DDE	second-order Cauchy problems on Y with delayed feedback control (with S.M. Verduyn Lunel), and
RE + DDE	models of structured populations (with O. Diekmann).

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[5] [V. Gils, Janssens, Kuznetsov, Visser, 2013], [Spek, V. Gils, Kuznetsov, 2019]

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