A class of abstract delay differential equations

in the light of suns and stars

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Outline

Introduction and purpose

The sun-star duality structure

Abstract DDEs as integral equations

Forthcoming work and comments

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Abstract delay differential equations

We are interested in the initial value problem

$$\dot{x}(t) = Bx(t) + F(x_t), \qquad t \geq 0, \tag{DDE}$$

$$x_0 = \varphi \in X, \tag{IC}$$

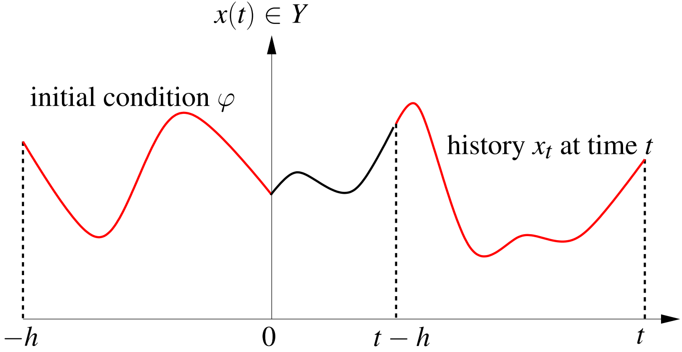
where

- Y is an arbitrary real or complex Banach space,
- B generates a $\mathcal{C}_0$-semigroup of linear operators on Y,
- $X := C([-h, 0], Y)$ is the state space,
- $x_t \in X$ is the history at time $t \geq 0$,

$$x_t(\theta) := x(t + \theta), \qquad \forall\, \theta \in [-h, 0],$$

- $F : X \to Y$ is continuous and may be nonlinear.

We call (DDE) an abstract DDE with initial condition (IC).



$x(t) \in Y$
initial condition φ
history x_t at time t
$-h$
0
$t-h$
t

Examples of existing work

There is a substantial literature on the semilinear problem:

Using formal dualities [Hale, Verduyn Lunel]:

- [Travis and Webb, 1974],
- [Wu, 1996],
- [Faria, Huang and Wu, 2002],
- [Faria, 2006].

These works assume that S is compact.
If $\dim Y = \infty$ then compactness of S excludes $B = 0$.

Purely linear theory in spaces of continuous or integrable functions:

- [Engel and Nagel, 2000],
- [Bátkai and Piazzera, 2005].

Aim of the present work

1. Establish one-to-one correspondence between (DDE, IC) and

$$u(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0^{\odot\star}(t-\tau)G(u(\tau))\,d\tau \qquad \text{(AIE)}$$

   for an appropriate $\mathcal{C}_0$-semigroup T_0 on X and a continuous nonlinear perturbation $G : X \to X^{\odot\star}$.

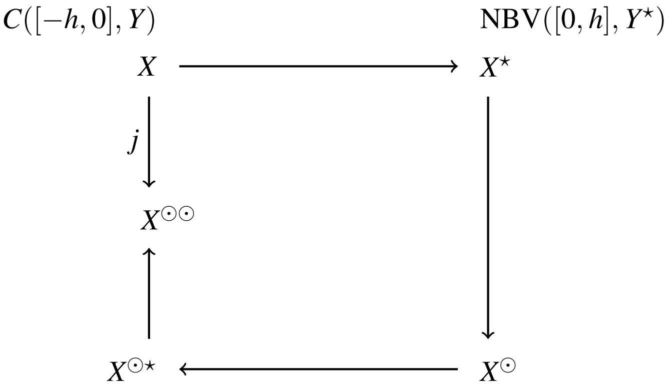
2. Study bounded linear perturbations of an arbitrary $\mathcal{C}_0$-semigroup on an arbitrary non-sun-reflexive Banach space.

3. Study nonlinear Lipschitz continuous perturbations and verify existence and properties of local invariant manifolds.

4. Apply general results to the particular class of abstract DDEs.

Outline

The sun-star skeleton



Now specify T_0 on X and find a representation of the sun dual $X^{\odot}$.

Specification of T_0 on X

Inspired by classical DDEs, consider the 'trivial' initial value problem

$$\begin{cases} \dot{x}(t) = Bx(t), & t \geq 0, \\ x_0 = \varphi \in X, \end{cases}$$

for (DDE) with $F = 0$.

It has the explicit mild solution $x^\varphi : [-h, \infty) \to Y$ given by

$$x_0^\varphi = \varphi, \qquad x^\varphi(t) = S(t)\varphi(0), \qquad t \geq 0.$$

Define the strongly continuous shift semigroup T_0 on X by

$$T_0(t)\varphi := x_t^\varphi,$$

for all $\varphi \in X$ and $t \geq 0$.

A representation theorem for $X^{\odot}$

Define $\chi_0 : [0,h] \to \{0,1\}$ by $\chi_0(t) = \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases}$

Theorem

The maximal subspace of strong continuity of $T_0^{\star}$ is

$$X^{\odot} = \Big\{ f : [0,h] \to Y^{\star} \ : \ \textit{there exist } y^{\odot} \in Y^{\odot} \textit{ and } g \in L^1([0,h], Y^{\star}) \\ \textit{such that } f(t) = \chi_0(t) y^{\odot} + \int_0^t g(s)\, ds \Big\},$$

and $\iota : Y^{\odot} \times L^1([0,h], Y^{\star}) \to X^{\odot}$ defined by

$$\iota(y^{\odot}, g)(t) := \chi_0(t) y^{\odot} + \int_0^t g(s)\, ds, \qquad \forall\, t \in [0,h],$$

is an isometric isomorphism.

The proof uses properties of the bilinear Riemann-Stieltjes integral.

Proof.

1. Denote by $E \subseteq X^\star$ the candidate subspace for $X^\odot$.
2. Find a representation for $T_0^\star$ on E.
3. For arbitrary $f \in E$ show that

$$|\langle \varphi, T_0^\star(t)f\rangle - \langle \varphi, f\rangle| \to 0, \qquad t \downarrow 0,$$

   uniformly for $\|\varphi\| = 1$. This proves $E \subseteq X^\odot$.
4. E is closed because ι is a linear isometry into $X^\star$.
5. Show that $R(\lambda, A_0^\star)$ maps $X^\star$ into E.
6. Take closures on both sides of $\mathcal{D}(A_0^\star) \subseteq E$. □

The resolvent trick was inspired by [Greiner and Van Neerven, 1992].

From now on, we identify $X^{\odot} \simeq Y^{\odot} \times L^1([0,h], Y^{\star})$.

Theorem

The duality pairing between $\varphi \in X$ and $\varphi^{\odot} = (y^{\odot}, g) \in X^{\odot}$ is

$$\langle \varphi, \varphi^{\odot} \rangle = \langle \varphi(0), y^{\odot} \rangle + \int_0^h \langle \varphi(-\theta), g(\theta) \rangle \, d\theta.$$

For the action of $T_0^{\odot}$ on $\varphi^{\odot} = (y^{\odot}, g) \in X^{\odot}$ we have

$$T_0^{\odot}(t)\varphi^{\odot} = (S^{\odot}(t)y^{\odot} + \int_0^{t \wedge h} S^{\star}(t-\theta)g(\theta)\, d\theta, T_1(t)g),$$

where T_1 is translation on $L^1([0,h], Y^{\star})$ and the integral is a weak* Lebesgue integral with values in $Y^{\odot}$.

Dressing the sun-star skeleton

T_0 on $C([-h,0],Y)$ $\qquad\qquad$ $\mathrm{NBV}([0,h],Y^\star)$

$$\begin{array}{ccc} X & \longrightarrow & X^\star \\ j \downarrow & & \downarrow \\ X^{\odot\odot} & & \\ \uparrow & & \\ X^{\odot\star} & \longleftarrow & X^{\odot} \end{array}$$

$Y^{\odot\star} \times [L^1([0,h],Y^\star)]^\star$ $\qquad\qquad$ $Y^{\odot} \times L^1([0,h],Y^\star)$

Comments on the sun-star dual $X^{\odot\star}$

We have the identification

$$[L^1([0,h],Y^\star)]^\star \simeq L^\infty([-h,0],Y^{\star\star})$$

if and only if $Y^{\star\star}$ has the Radon-Nikodým property.

So, in general we can only write

$$X^{\odot\star} \simeq Y^{\odot\star} \times [L^1([0,h],Y^\star)]^\star.$$

Still, $Y^{\odot\star} \times L^\infty([-h,0],Y^{\star\star})$ is isometrically embedded in $X^{\odot\star}$.
Relevant for computations, for example in bifurcation theory.

Outline

The lack of sun-reflexivity

If Y is not sun-reflexive for S then X is not sun-reflexive for T_0.

For $u : \mathbb{R}_+ \to X$ continuous, the weak* integral in (AIE)

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0^{\odot\star}(t-\tau)G(u(\tau))\, d\tau$$

takes values in $X^{\odot\odot}$. If X is not sun-reflexive for T_0 then

$$j(X) \subset X^{\odot\odot}$$

with strict inclusion, so it is not clear if j^{-1} can be applied.

The range of the weak* integral

Recall that we have proven that $X^{\odot\star} \simeq Y^{\odot\star} \times [L^1([0,h], Y^\star)]^\star$.
Inspired by classical DDEs, define

$$\ell : Y \to X^{\odot\star}, \qquad \ell y := (j_Y y, 0).$$

Proposition

Let $f : \mathbb{R}_+ \to Y$ be continuous and let $t \geq 0$. Then

$$\int_0^t T_0^{\odot\star}(t-\tau)\ell f(\tau)\, d\tau = j\psi,$$

where $\psi \in X$ is given by

$$\psi(\theta) := \int_0^{(t+\theta)^+} S(t-\tau+\theta) f(\tau)\, d\tau, \qquad \forall\, \theta \in [-h, 0].$$

So, the weak* integral takes values in the range of j.

Proof.

1. Observe that ℓ has a 'pre-adjoint'. If we define

$$\delta : X^{\odot} \to Y^{\odot}, \qquad \delta(y^{\odot}, g) := y^{\odot},$$

   then $\delta^{\star} : Y^{\odot\star} \to X^{\odot\star}$ and $\ell = \delta^{\star} j_Y$.
2. Recall that we found a representation for $T_0^{\odot}$.
3. Use it to represent $T_0^{\odot\star}$ on the range of ℓ.
4. Check that

$$\langle \varphi^{\odot}, \int_0^t T_0^{\odot\star}(t-\tau)\ell f(\tau)\, d\tau \rangle = \langle \psi, \varphi^{\odot} \rangle, \qquad \forall\, \varphi^{\odot} \in X^{\odot}$$

   also using the duality pairing between X and $X^{\odot}$. □

Corollary

Let T_0 be the shift semigroup, $F : X \to Y$ Lipschitz continuous and $G := \ell \circ F$. For every initial condition $\varphi \in X$ there exists a unique continuous function $u : \mathbb{R}_+ \to X$ that satisfies

$$u(t) = T_0(t)\varphi + j^{-1} \int_0^t T_0^{\odot\star}(t-\tau)G(u(\tau))\,d\tau,$$

i.e. u is the global solution of (AIE).

As usual, a local Lipschitz condition gives local solutions that are unique in the maximal sense.

Correspondence between (DDE, IC) and (AIE)

We return to (DDE, IC),

$$\dot{x}(t) = Bx(t) + F(x_t), \qquad t \geq 0, \tag{DDE}$$

$$x_0 = \varphi \in X, \tag{IC}$$

to make the connection with (AIE).

Definition

A continuous function $x : [-h, \infty) \to Y$ that satisfies $x_0 = \varphi$ and

$$x(t) = S(t)\varphi(0) + \int_0^t S(t-\tau)F(x_\tau)\, d\tau, \qquad \forall\, t \geq 0,$$

is called a mild solution of (DDE, IC).

Any classical solution is a mild solution.
If $B = 0$ then any mild solution is a classical solution.

As a consequence of the earlier proposition on the range of the weak* integral, we arrive at:

Theorem

Let T_0 be the shift semigroup, $F : X \to Y$ continuous and $G := \ell \circ F$. Let $\varphi \in X$ be an initial condition.

1. *Suppose x is a mild solution of* (DDE, IC). *Define $u : \mathbb{R}_+ \to X$ by*

$$u(t) := x_t, \qquad \forall\, t \geq 0.$$

   Then u is a solution of (AIE).

2. *Suppose u is a solution of* (AIE). *Define $x : [-h, \infty) \to Y$ by*

$$x_0 := \varphi, \qquad x(t) := u(t)(0), \qquad t \geq 0.$$

   Then x is a mild solution of (DDE, IC).

Outline

Forthcoming work

Given this correspondence, standard results from ‘the book’, such as

- the principle of linearized stability,
- existence and smoothness of local invariant manifolds,
- local bifurcation theorems,

are expected to remain valid for abstract DDEs.

To make this rigorous, we need to check in detail

- what happens without sun-reflexivity,
- if weaker conditions are sufficient. (Yes, so far.)

Maybe this could also be relevant for twin semigroups?
[Diekmann and Verduyn Lunel, forthcoming, 2019]

Aim of the forthcoming work

1. Establish one-to-one correspondence between (DDE, IC) and

$$u(t) = T_0(t)\varphi + j^{-1}\int_0^t T_0^{\odot\star}(t-\tau)G(u(\tau))\,d\tau \qquad \text{(AIE)}$$

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Comments

Motivating example

In [Van Gils, Janssens, Kuznetsov and Visser, 2013] an abstract DDE with $B = 0$ and non-reflexive Y was considered. There,

- existence of a smooth local center manifold,
- differentiability of solutions of (AIE) that lie on it,

were already assumed to hold in the non-sun-reflexive case.

Regularity of S

We will require more than just strong continuity from S.
If S is immediately norm continuous, then T is eventually norm continuous and

$$s(A) = \omega_0(T).$$

For further reading

- O. Diekmann and M. Gyllenberg.
  Abstract delay equations inspired by population dynamics.
  In *Functional Analysis and Evolution Equations*, pages 187–200.
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